B.A/B.Sc 6th Semester (Honours) Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMH6 CC 13
(Metric Spaces and Complex Analysis)

Time:3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: 6×5 = 30

(a) Prove that a compact metric space is complete. 5

(b) Prove that a metric space (X,d) is disconnected if and only if there exists a continuous function $f: X \to \{0,1\}$, which is onto. 5

(c) (i) Prove that for any two distinct points x, y in a metric space (X,d) there exist two open sets U and V in X containing x and y respectively such that $U \cap V = \phi$

(ii) Give an example with justification of a non-trivial map f from a metric space onto itself which have infinitely many fixed points. 3+2

(d) Let (X,d) and (Y,ρ) be two metric spaces. Prove that a mapping $f: X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subseteq X$. 5

(e) (i) Evaluate $\int_{\gamma} \frac{e^{2z}}{(z+1)^4} dz$, where $\gamma = \{z \in \mathbb{C} : |z| = 3\}$

(ii) Examine the convergence of the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, where $z \in \mathbb{C}$. 3+2

(f) Show that $u(x,y) = e^x(x\cos y - y\sin y)$ satisfies Laplace equation and find its harmonic conjugate function $v(x,y)$ so that $f(z) = u(x,y) + iv(x,y)$ is analytic, where $z \in \mathbb{C}$ 2+3

(g) State and prove Liouville's Theorem. 1+4

(h) (i) Let (X,d) be a metric space and A be a non-empty subset of X. Suppose $f: X \to \mathbb{R}$ be given by $f(x) = d(x,A), x \in X$. Show that $f(x) = 0$ if and only if $x \in \overline{A}$.

(ii) Prove that $\arg z - \arg(-z) = \pm\pi$ according as $\arg z$ is positive or negative, where $z \in \mathbb{C}$ 3+2

2. Answer any three questions: 10×3 = 30

(a) (i) State and prove Banach Contraction Principle.

(ii) Let (X,ρ) and (Y,σ) be two metric spaces and $f:(X,\rho)\to(Y,\sigma)$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in (X,ρ) then show that $\{f(x_n)\}$ is also a Cauchy sequence in (Y,σ). (1+5)+ 4

(b) (i) Prove that a metric space (X,ρ) is complete if for every descending sequence of non-empty closed sets $\{C_n\}$ in (X,ρ) with $diam(C_n)\to 0$ as $n\to\infty$, the intersection $C=\bigcap_{n=1}^{\infty} C_n$ consists of single point only.

(ii) If A is a connected set in a metric space (X,ρ) and $A\subseteq B\subseteq \overline{A}$ then show that B is connected and hence show that $\overline{A}$ is connected. 5+ (3+2)

(c) (i) Let (X,ρ) and (Y,σ) be two metric spaces and $f:(X,\rho)\to(Y,\sigma)$ be a continuous function. If $K\subseteq X$ is a compact set in (X,ρ) then prove that $f(K)$ is a compact set in (Y,σ). Hence show that a surjection $f:[a,b]\to C\,(C\subseteq\mathbb{R})$, where C is not closed in $\mathbb{R}$, cannot be continuous.

(ii) Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function and suppose $|f'(x)|<1$ on $\mathbb{R}$. Show that f is a contraction map on $\mathbb{R}$. (4+3)+3

(d) (i) Let $f=u+iv$ be a differentiable function on a region $G\subset\mathbb{C}$. If $f(G)$ is a path of a circle in $\mathbb{C}$, show that f is constant.

(ii) Prove that every non-constant polynomial with complex coefficients has a zero in the field of complex numbers. 5+5

(e) (i) Find the Laurent series expansion of $f(z)=\dfrac{1}{z(z-1)(z-2)}$ in $2<|z|<\infty$.

(ii) By contour integration show that $\int_0^{\infty}\dfrac{\sin x}{x}dx=\dfrac{\pi}{2}$. 4+6