B.A/B.Sc 3rd Semester (Honours) Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMH3CC06 (Group Theory-1)

Time: 3 Hours — Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: $6\times5 = 30$

(a) Prove that $\mathbb{Z}_3 \otimes \mathbb{Z}_5 \cong \mathbb{Z}_{15}$. [5]

(b) In S_4, find a cyclic subgroup of order 4 and a noncyclic subgroup of order 4. [5]

(c) (i) Let H be a subgroup of the group G of index 2.Prove that H is normal in G. [3]

(ii) Prove that $\mathbb{Z}$ under addition is not isomorphic to $\mathbb{Q}$ under addition. [2]

(d) Show that A_5 has no normal subgroup. [5]

(e) Determine the number of cyclic subgroups of order 10 in $Z_{100}\otimes Z_{25}$. [5]

(f) Prove that a cyclic group of finite order *n* has one and only one subgroup of order *d* for every positive divisor *d* of *n*. [5]

(g) Let G be a cyclic group of order 6 generated by x. Let H, K be the subgroups generated by x^2, x^3 respectively. Prove that $|H|=3$, $|K|=2$, $G=HK$, and that $H\cap K=\{e\}$. [5]

(h) Show that all proper subgroups of the quaternion group Q_8 are cyclic. [5]

2. Answer any three questions: $3\times10 = 30$

(a) (i) If H is a subgroup of finite index in G, prove that there is only a finite of distinct subgroups in G of the form aHa^{-1}. [5]

(ii) Does there exist subgroup H of $\mathbb{Z}$ other than $n\mathbb{Z}$? Justify your answer. [5]

(b) (i) Let G be a finite group whose order is not divisible by 3. Suppose that $(ab)^3=a^3b^3$ for all x in G. Prove that G must be abelian. [5]

(ii) How many generators does a cyclic group of order n have? Justify your answer. [5]

(c) (i) Suppose H is the only subgroup of order |H| in the finite group G. Prove that H is normal subgroup. [5]

(ii) Prove that a group of order 9 is abelian. [5]

(d) (i) Suppose that N and M are two normal subgroups of G and that $N\cap M=\{e\}$.Show that for any n in N and m in M, nm=mn. [5]

(ii) Let H be group of order n which is also a homomorphic image of G. Let k >1 be a divisor of |G| such that gcd (k,n)=1. Then show that G is not simple. [5]

(e) (i) Let ϕ: $G\to G^{/}$ be a homomorphism of a group G onto a group $G^{/}$ and θ be the natural homomorphism of G onto G/H where H=$ker\phi$. Prove that there exists [5] an isomorphism ψ: G/H→$G^{/}$such that $\phi=\psi\theta$.

(ii) Let G={$\begin{bmatrix} a & a \\ a & a \end{bmatrix}$| a in $\mathbb{R}$ a$\neq$ 0}. Show that G is a group under matrix multiplication. [5]